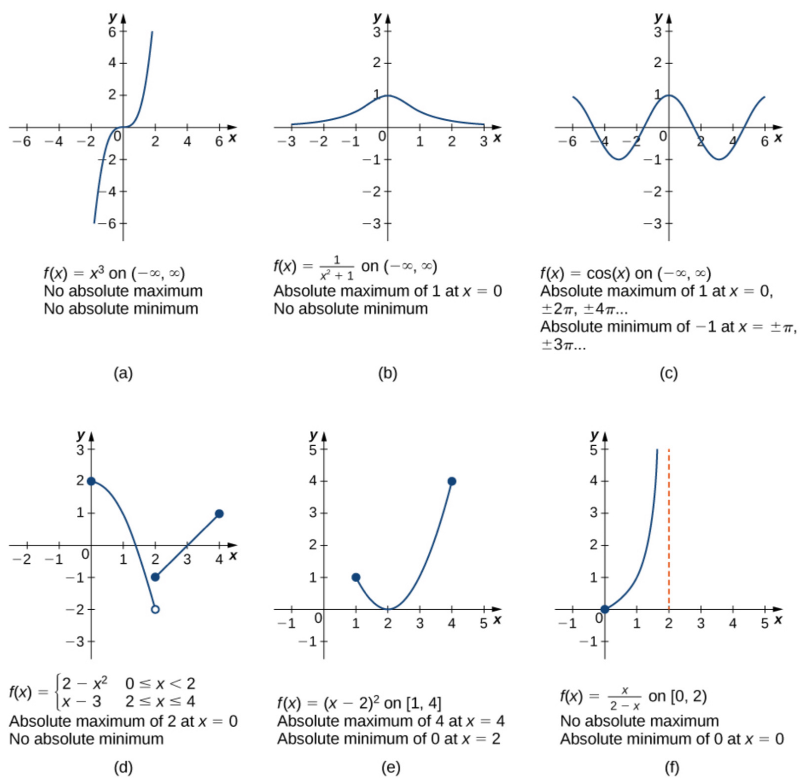
Maxima & Minima

Absolute Maxima or Minima:

Let $f: I \to \mathbb{R}$ (reals) be a function. Let $c \in I$.

f has $\boxed{\text{absolute maxima}}$ if $f(c) \geq f(x)$ for all $x \in I$.
f has $\boxed{\text{absolute minima}}$ if $f(c) \leq f(x)$ for all $x \in I$.

absolute extrema at $x = c$



$f(x) = x^3$ on $(-\infty, \infty)$
No absolute maximum
No absolute minimum

(a)

$f(x) = \frac{1}{x^2+1}$ on $(-\infty, \infty)$
Absolute maximum of 1 at $x = 0$
No absolute minimum

(b)

$f(x) = \cos(x)$ on $(-\infty, \infty)$
Absolute maximum of 1 at $x = 0, \pm 2\pi, \pm 4\pi \ldots$
Absolute minimum of -1 at $x = \pm\pi, \pm 3\pi \ldots$

(c)

$f(x) = \begin{cases} 2 - x^2 & 0 \leq x < 2 \\ x - 3 & 2 \leq x \leq 4 \end{cases}$
Absolute maximum of 2 at $x = 0$
No absolute minimum

(d)

$f(x) = (x-2)^2$ on $[1, 4]$
Absolute maximum of 4 at $x = 4$
Absolute minimum of 0 at $x = 2$

(e)

$f(x) = \frac{x}{2-x}$ on $[0, 2)$
No absolute maximum
Absolute minimum of 0 at $x = 0$

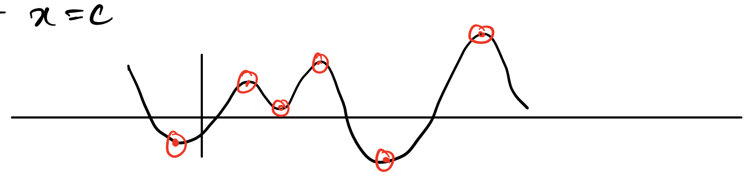
(f)

Local Maxima & Minima:

Let $f: I \to \mathbb{R}$ be a function. Let J be a subset of I such that c is in J.

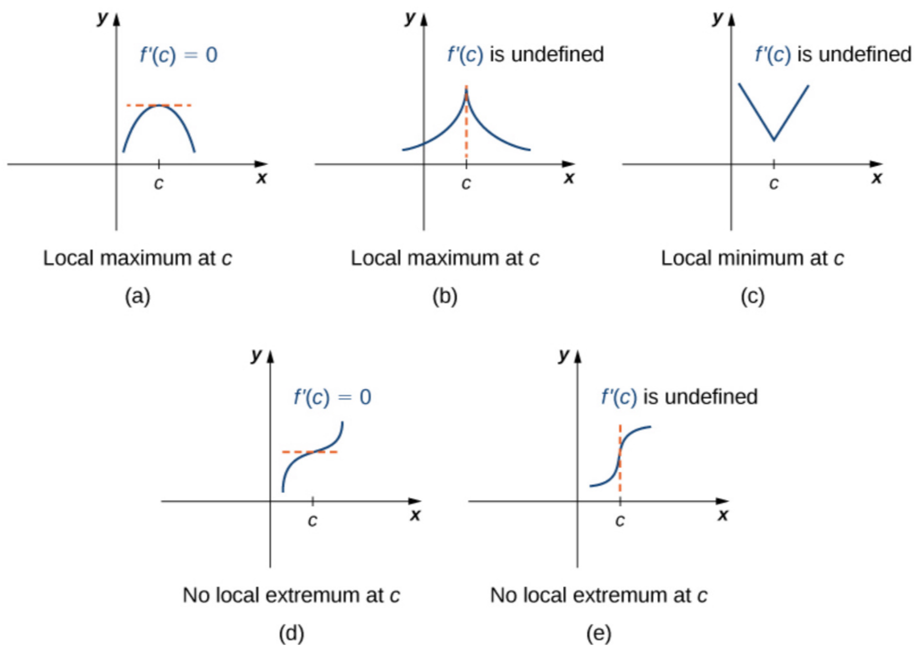
f has $\boxed{\text{local maxima}}$ if $f(c) \geq f(x)$ for all $x \in J$.
f has $\boxed{\text{local minima}}$ if $f(c) \leq f(x)$ for all $x \in J$.

local extrema at $x = c$



<u>Critical Point</u> : f has a critical point at $x = c$ if $f'(c) = 0$ or $f'(c)$ undefined.

Format's Theorem : A function f has local extremum at $x = c$ if f is differentiable at $x = c$ & $f'(c) = 0$.



Eg. $f(x) = -x^2 + 3x - 2$, $x \in [1, 3]$

Note, this is closed interval. So, we need to examine the end points.

For extrema: $f'(x) = 0$ or undefined

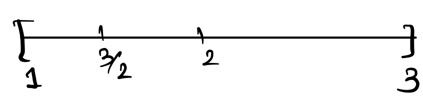
Now, $f'(x) = -2x + 3$

$$\Rightarrow -2x + 3 = 0 \rightsquigarrow 2x = 3 \rightsquigarrow x = \frac{3}{2}$$

At $x = \frac{3}{2}$, $\boxed{f(x) = 0.25} \rightarrow$ Absolute Maxima

$x = 1$, $f(x) = 0 \rightsquigarrow$ Local extrema

$x = 3$, $\boxed{f(x) = -2} \rightarrow$ Absolute Minima



Detection of Local Extrema:

Say $x=c$ is a critical point of f, ie, $f'(c)=0$.

Case 1: $f'(x)$ is +ve for $x<c$ & $f'(x)$ is -ve for $x>c$

$\Rightarrow f(c)$ is a local maxima.

Case 2: $f'(x)$ is -ve for $x<c$ & $f'(x)$ is +ve for $x>c$.

$\Rightarrow f(c)$ is a local minima.

Case 3: $f'(x)$ has the same sign on both side of $x=c$

$\Rightarrow f(c)$ is $\underbrace{\text{neither a local maxima or minima}}_{\text{saddle point}}$.